Int. J. Production Economics 155 (2014) 82–90

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Economic Order Quantities in production: From Harris to Economic Lot Scheduling Problems

Martin Holmbom [1], Anders Segerstedt *

Industrial logistics, ETS, Luleå university of Technology, SE-971 87 Luleå, Sweden

ARTICLE INFO

Article history:
Received 30 May 2013
Accepted 31 March 2014
Available online 8 April 2014

Keywords:
Economic Order Quantities
Economic Production Lots
Economic Lot Scheduling Problems

ABSTRACT

This article provides a short historical overview from Harris and his Economic Order Quantity (EOQ) formula to the Economic Lot Scheduling Problem (ELSP). The aim is to describe the development of the ELSP field from the EOQ formula to the advanced methods of today in a manner that suits master and graduate students. The article shows the complexities, difficulties and possibilities of scheduling and producing several different items in a single production facility with constrained capacity. The items have different demand, cost, operation time and set-up time. Set-up time consumes capacity and makes the scheduling more complicated. Idle time makes the scheduling easier but is bad from a practical point of view since it creates unnecessary costs due to low utilisation of the facility. A heuristic solution method is used on a small numerical example to illustrate different solution approaches. The solution method creates a detailed schedule and estimates the correct set-up and inventory holding cost even if the facility works close to its capacity.

1. Introduction

The *Economic Order Quantity* (EOQ) has been a known concept in the manufacturing industry for 100 years. Over the years numerous methods for calculating economic order quantities have been developed and these methods have become increasingly advanced as their creators have aimed to solve more intricate problems. One of these problems is the *Economic Lot Scheduling Problem* (ELSP) which is about calculating economic order quantities and constructing a feasible production schedule for several products that are produced in the same machine or production facility. This may sound like a trivial problem but it is indeed intricate and even classified as NP-hard (Gallego and Shaw, 1997). Due to the complexity of the problem, the reader of ELSP literature must often be well-read in mathematics to fully understand the solution methods and apply them to real cases. Therefore, this article aims to describe the development of the ELSP field from the EOQ formula to the advanced methods of today in a manner that suits master and graduate students. The article focuses on the development of the EOQ formula and the three main approaches in previous literature that most ELSP methods are based upon, the *common cycle solution*, the *basic period approach* and the *extended basic period approach*. To illustrate these approaches, we solve a small numerical example with our own heuristic method. Using this example, we will show and argue where the research frontier concerning deterministic ELSP is. We will also show that the traditional inventory holding cost approximation may not always hold.

The ELSP problem is found in countless practical applications, e.g. milling of gear houses, painting of metal rolls, welding of rear axles, painting of truck components, moulding of brackets, paper production, etc.; reaching from process industries with more or less continuous flow to work shops. In many cases it is financially beneficial for companies to purchase and run *one* flexible high-speed machine capable of processing many types of items, compared to purchase and run many dedicated machines (Gallego and Roundy, 1992; Segerstedt, 1999). Hence, many machines produce more than one item. Typically, such machines are capable only of processing one type of item at a time, and a set-up is usually required each time the production switch from one type to another (Gallego and Roundy, 1992). Thus, the dominant characteristics of a single-machine multi-item ELSP system are the following: multiple items are processed; only one item can be produced at a time; the machine has limited but sufficient capacity; a set-up is required between the processing of different items; items may differ in cost structure and require different machine capacity; backorders are not allowed; Item demand rates are deterministic and constant over time; set-up and operation times are deterministic and constant over time; set-up costs and set-up times are independent of production sequence; inventory holding costs are determined by the value of stocks held.

* Corresponding author. Tel.: +46 920491212.
E-mail addresses: Martin.Holmbom@ltu.se (M. Holmbom), Anders.Segerstedt@ltu.se (A. Segerstedt).
[1] Tel.: +46 920492427.

http://dx.doi.org/10.1016/j.ijpe.2014.03.024

Bomberger (1966) presents a problem and a solution method from the characteristics of a metal stamping facility producing different stampings on the same press line. Bomberger's 10-items problem became a "milestone" concerning ELSP; the problem has been the test example(s) new suggestions and methods have tried to solve and compete with. Because of that we present the problem and a totally scheduled solution of it, to the best of our knowledge not presented in the same way before.

The article has the following outline: Section 2 provides a short resume on the early history of EOQ and the Economic Production Lot (EPL). Section 3 introduces our small numerical example. Sections 4–6 treat the common cycle solution, the basic period approach and the extended basic period approach respectively. In Section 7 we present the Bomberger problem and provide a complete scheduled solution to it, followed by a short review of the literature from Bomberger to present time. Finally, in Section 8 we summarise the article and discuss ELSP from a practical point of view.

2. The early history of EOQ and Economic Production Lot (EPL)

In 1913, Ford Whitman Harris published the first ideas about the EOQ in "Factory, The Magazine of Management". Harris (1913) uses the following notations: M=the number of units used per month; C=the cost of a unit ($); S=the set-up cost of an order ($); T=the manufacturing interval in months; I=the unit charge for interest and depreciation on stock; X=the unknown order size, or lot size, that is best from an economic perspective. Harris assumes that the annual interest and depreciation cost is 10%, and he formulates a total-cost function that, in contrast to most textbooks of today, calculates the cost per unit instead of the cost per time interval (day or year) and includes the set-up cost of the average stock ($S/2$)

$$\frac{1}{240M}(CX+S)+\frac{S}{X}+C. \tag{1}$$

Harris states that finding the order size X that minimises Eq. (1) involves higher mathematics. Harris further continues that "suffice it to say that the value for X that will give the minimum value [...] reduces to the square root of 240MS divided by C"; an equation now known as the classic EOQ formula

$$\sqrt{\frac{240MS}{C}}=\sqrt{\frac{2\times 12MS}{0.1C}}. \tag{2}$$

Erlenkotter (1990) presents a compilation of early EOQ literature and describes Harris' complete career from a production engineer to a patent lawyer and a founder of a law firm. Erlenkotter (1990) explains why the EOQ formula also is known as the *Wilson lot size formula* and *Camp's formula*. Wilson (1934) made the EOQ formula famous when he created and sold an inventory control scheme based on Harris' ideas. However, he acted according to the old tradition and did not cite earlier work. (In the early 1970s the formula was known as the Wilson-formula in Sweden.) Moreover, in a handbook on cost and production, Alford (1934) referenced Camp (1922) for a general formula to determine the EOQ. In the first edition of the "Industrial Engineering Handbook" published by McGraw-Hill (Maynard, 1956, pp. 8–182), W.W. Hannon correctly identified Harris' work and Taft (1918), who extends Harris EOQ formula by incorporating a finite production rate. But in the next edition of the same handbook another author attributed the EOQ formula to Camp (1922). For further information about these confusing citations up to the 1970s the reader is referred to Erlenkotter (1990).

We now introduce our own notations that we will use in the rest of the article d=demand rate of the item, in units per day; h=inventory holding cost of the item, in money units per unit and day; A=set-up cost or order cost, in money units (a fixed cost per order or production lot); q=order quantity of the item.

Then, the cost per time unit can be written as a function of q

$$C(q)=\frac{dA}{q}+h\frac{q}{2}. \tag{3}$$

From Eq. (3) we can see that the cost per unit is

$$C_q(q)=\frac{A}{q}+h\frac{q}{2}\frac{1}{d}. \tag{4}$$

The resemblances to Harris' equation are $h=0.1C$, $d=12M$ and $A=S$. Grubbstrom (1995) points out that Eq. (4), through algebraic rearrangements, can be written as

$$C_q(q)=\frac{h}{2dq}\left(q-\sqrt{\frac{2dA}{h}}\right)^2+\sqrt{\frac{2Ah}{d}}. \tag{5}$$

In Eq. (5), the first term is always positive, except when the expression inside the square is zero, which coincides with the minimum point of the cost function. The second term, which is independent of q and thus constant, corresponds to the lowest possible cost per unit of the order quantity. Hence, both differential calculus on Eq. (3) and the algebraic expression in Eq. (5) can be used to derive the EOQ formula

$$q^*=EOQ=\sqrt{\frac{2dA}{h}}. \tag{6}$$

It is not known how Harris first derived the EOQ formula but it is an interesting question due to the fact that Harris had limited education in mathematics (cf. Erlenkotter (1990)).

Up to now we have assumed that all replenishments happen instantly, i.e. the production rate is infinite. However, if the production rate is finite, the assumption of the average inventory ($q/2$) changes since the maximum inventory become less than the order quantity. To handle this, we let κ be the production rate in units per day and assume that the production is on-going during the time t. Then, $q=\kappa t$, and since the demand is constant at all time (even during the production) the maximum inventory is $(\kappa-d)t$, which is smaller than the order quantity. $t=q/\kappa$ so the maximum inventory can be written $q(\kappa-d)/\kappa$. The average inventory is half the maximum inventory and hence the cost function, in money units per time unit, is

$$C(q)=\frac{dA}{q}+h\frac{q}{2}\frac{(\kappa-d)}{\kappa}. \tag{7}$$

The order quantity corresponding to the minimum cost is derived through differential calculus on Eq. (7)

$$q^*=EPL=\sqrt{\frac{2dA}{h}\frac{\kappa}{\kappa-d}}=\sqrt{\frac{2dA}{h}\frac{1}{1-d/\kappa}} \tag{8}$$

This quantity is usually called the *Economic Production Lot* (EPL) and the formula, Eq. (8), was according to Erlenkotter (1990) first derived in Taft (1918). Like the EOQ formula, the EPL formula is well-known and commonly described in textbooks for undergraduate and graduate students in operations management and logistics. Different terms for EPL are used by different authors; e.g. Buffa (1969) uses minimum cost Production Order Quantity, and Silver et al. (1998) use Economic Production Quantity.

3. A small numerical example

We will now introduce a small numerical example with a single machine and multiple items that we will use throughout this article. Suppose that we have a machine that produces three different items, A, B and C, with production and demand data as in Table 1. The machine can only produce one item at a time and there is a set-up time that precedes every start of a new item. The capacity of the machine, i.e. the number of items that can be produced per day, depends on the number of working hours available. In this example, we assume that there are 14 working hours available per day, which means that we can produce 84 pieces of A in one day if we do not make any set-ups. (An index i is added to distinguish the items. An operation time, o_i, in production time (days) per produced item i, is also added so that $o_i = 1/\kappa_i$.) All notations used in the following calculations are gathered in Table 2.

To calculate the EPL of the items we apply Eq. (8) on the data presented in Table 1; $q_A^* = \sqrt{(2\times 10.5 \times 500/0.2174)(1/1-(0.0119\times 10.5))} \approx 235$, $q_B^* = 246$, and $q_C^* = 256$, and according to Eq. (7) the total cost is $\sum_i((d_iA_i/q_i^*)+h_i(q_i^*/2)(1-o_id_i))=299.40$ money units per day. This solution is a lower bound, i.e. the cost cannot be reduced any further. But, as the set-up time is not considered in the EPL formula, it is not certain that the order quantities will cover the demand during the total time of both production and set-up. However, in this particular example the demand for 230 days can be produced during less than 192 days with set-ups included.

Rogers (1958) discusses the problem of inventory control in a single-machine multi-item system and applies the EPL formula to the items individually. According to his conclusions, it is usually impossible to construct a feasible production schedule from these calculated order quantities, since two or more items would have to be produced at the same time to avoid demand shortages. The problem with such *interference* between the items is also discussed and illustrated by Brander (2006). Hadley and Whitin (1963, p. 54) state that "It is permissible to study each item individually only as long as there are no interactions among the items", and that "There can be many sorts of interactions among the items". They mention items competing for floor space and investments in inventory. Hence, the EOQ and EPL formulas, which only consider one product at the time, are limited from a practical point of view.

To calculate economic order quantities in a single-machine multi-item system and avoid interferences between the items, we need to handle the items jointly. Hence, consider a machine where several items are produced ($i = 1, 2, \ldots, N$) and assume finite production rates and instant set-ups. Then, for a common time interval, T, during which all items are produced *once*, the cost per time unit is

$$C(T) = \sum_{i=1}^{N}\left(\overbrace{\frac{A_i}{T}}^{\text{set-up cost}} + \overbrace{h_i\frac{d_iT}{2}\frac{(\kappa_i-d_i)}{\kappa_i}}^{\text{inventory holding cost}}\right) = \sum_{i=1}^{N}\left(\frac{A_i}{T}+h_i\frac{d_iT}{2}(1-o_id_i)\right). \tag{9}$$

The optimal T that corresponds to the lowest cost is found through differential calculus on Eq. (9)

$$T^* = \sqrt{\frac{2\sum_{i=1}^{N}A_i}{\sum_{i=1}^{N}h_id_i(1-o_id_i)}}. \tag{10}$$

Eqs. (8) and (9) are often presented in textbooks to introduce a discussion of cyclic policies (e.g. Nahmias (2009) and Segerstedt (2009)). However, those equations only find a common cycle length that minimises the set-up cost and inventory holding cost, and do not consider the possibility that the available capacity may be too small to satisfy the required demand.

Table 1
Data for the numerical example.

		A	*B*	*C*
Demand rate (units/day)	d_i	10.5	30	17
Machine capacity (min/unit)	o_i'	10	8	20
Set-up time (min)	s_i'	30	60	30
Set-up cost (money units, for every set-up)	A_i	500	500	1000
Cost/value (money units/unit)	c_i	500	1600	2000
(money units/day and unit)	h_i	0.2174	0.6957	0.8696
Shift factor (min/day)	K	840		
Interest rate (%/year)	r	10		
Production days (days/year)	D	230		
Production rate (units/day)	$\kappa_i = K/o_i'$	84	105	42
Actual capacity (days/unit)	$o_i = 1/\kappa_i$	0.0119	0.0095	0.0238
Set-up time (days)	$s_i = s_i'/K$	0.0357	0.0714	0.0357
Inv. holding cost (money units/day and unit)	$h_i = c_ir/D$	0.2174	0.6957	0.8696

Table 2
The notations used in this article.

d_i	Demand rate for item i, in units per day; i=1,2,…,N
h_i	Inventory holding cost of item i, in money units per unit and day
A_i	Set-up cost for item i, in money units per production lot
q_i	Order quantity for replenishment of item i, in units
κ_i	Production rate for item i, in units per day
s_i	Set-up time of item i, in days per production lot
o_i	$=1/\kappa_i$, Operation time of item i, in days per unit
T	Production cycle time, in days (time interval in which all items are produced at least once)
f_i	Frequency, the number of times that item i is produced during a production cycle T
$T_{\inf}$	The shortest possible production cycle time in days, in which all items can be produced with frequencies all equal to one
$T_{\min}$	The shortest possible production cycle time in days, in which all items can be produced with the chosen frequencies f_i ($T_{\inf} \le T_{\min}$)
$C(\mathbf{f},T)$	Total cost per day, in money units (depending on chosen frequencies and time interval)
t_{ij}	Adjusted early start of item i in period j, in days, before the inventory reaches zero
i_i	Current inventory of item i
$\hat{f}$	The highest frequency used; $\max_i(f_i)$

4. The common cycle solution

To take both capacity constraints and interferences between the items into account, we assume that the items A, B and C are produced during a common cycle that is repeated successively. Each time the cycle is finished it immediately starts over again. Thus, the cycle time, T, must be large enough to cover the set-up and operation times for all items of the demand during the cycle

$$\sum_i (s_i + o_i d_i T) \le T. \tag{11}$$

Eq. (11) can be rearranged to find the shortest possible cycle time

$$T_{\text{inf}} = \frac{\sum_i s_i}{1 - \sum_i o_i d_i}. \tag{12}$$

Eqs. (10) and (12) are combined to find the cycle time that corresponds to the lowest cost

$$T^* = \max\left(\sqrt{\frac{2\sum_{i=1}^N A_i}{\sum_{i=1}^N h_i d_i (1 - o_i d_i)}}, T_{\text{inf}} \right) \tag{13}$$

and the optimal order quantities are calculated from $q_i^* = d_i T^*$. This solution method was showed by Hanssmann (1962) and is known as the *common cycle solution*. The solution method guaranties feasible solutions in the sense that the production can be scheduled without interferences and that the machine capacity is sufficient. The common cycle solution to the numerical example is presented in Table 3 and illustrated in Fig. 1. In Table 3 we have introduced a frequency, f_i, which is the number of times that the item is produced during T. We will come back to the use of f_i later in this article.

According to Eqs. (12) and (13) $T_{\text{inf}} = 0.77$ and $T^* = 12.475$. The total set-up cost per day is calculated from $\sum_i f_i A_i / T^*$ and the total inventory holding cost per day is calculated from $\sum_i h_i (1 - o_i d_i) d_i T^* / (2 f_i)$. The total cost is 320.64 money units per day. During each cycle the machine is idle for $12.475 - 10.315 = 2.16$ days. The idle time is necessary since the output per day would exceed the demand rate if the machine would run non-stop, and the inventory would therefore increase with every cycle. However, from a practical point of view it seems strange to let the machine be idle for approximately 1/6 of the total time, since facilities and workforce cost money even during idle time (cf. Brander and Segerstedt (2009)). When there is much idle time it could be a good idea to reduce the number of shifts per day and thus reduce the available production time. But in this particular example it is not possible to reduce the number of shifts from two to one ($K = 420$), since the capacity would become too small to meet the demand, and s_i and o_i would increase to such extent that no positive T could satisfy Eq. (11).

Table 3
The common cycle solution of the numerical example.

		A	B	C	Σ
(1)	Frequency: f_i	1	1	1	
(2)	Set-up time: $f_i s_i$	0.0357	0.0714	0.0357	0.1428
(3)	Operation time: $o_i d_i T^*$	1.559	3.564	5.049	10.172
(4)	(2)+(3) Σ	1.595	3.635	5.085	10.315
(5)	Set-up cost/day	40.08	40.08	80.16	160.32
(6)	Inv. holding cost/day	12.46	92.98	54.88	160.32
(7)	(5)+(6) Σ				320.64
(8)	$q_i^* = d_i T^* / f_i$	131.0	374.2	212.1	
(9)	Ratio: (5)/(6)	3.22	0.43	1.46	
(10)	1/Ratio: 1/(9)	0.31	2.32	0.68	

5. The basic period approach

Hanssmann's article about the common cycle solution from 1962 was followed by a large amount of articles that treated similar problems, which we now know as ELSPs. Probably the most well-known ELSP publication was made by Earl E. Bomberger in 1966 when he presented his 10-item problem instance that has been cited extensively in the literature ever since. Bomberger (1966) uses dynamic programming to calculate order quantities, and he introduces an item-specific cycle time, T_i, which is the time from one production start of the ith item to the next production start of that item. Thus, the production of item i is repeated every T_i units of time. Bomberger constrains the cycle times T_i to be integer multiples k_i of a *basic period*, T_b, so that $T_i = k_i T_b$, and he restricts the basic period to be large enough to accommodate production of all items once. Bomberger's extension of the common cycle solution is called the *basic period approach*, and it allows the items to be produced with different *frequencies*.

To illustrate the basic period approach we return to our numerical example (in Table 1). We use the method presented in Holmbom et al. (2013). The main principle of that method was first developed by Segerstedt (1999). The principle is to reduce the total cost by finding an even balance between the set-up cost and the inventory holding cost of each item. Thus, the ratio between set-up cost and inventory holding cost of each item should be as close to 1 as possible. By making the ratios close to "1" we can expect to find a solution close to the lower bound solution (calculated from Taft's EPL formula) in which the ratio is equal to 1 for all items.

The frequency, f_i, was introduced in Table 3, where we defined it as the number of times that an item is produced during the cycle time T. If we look at the common cycle solution in Table 3 we can see that the largest ratio or inverted ratio (lines 9 and 10 in Table 3) belongs to item A. Since the ratio of item A is large, item A would probably benefit from less frequent production than items B and C, which would increase the order quantity of item A and hence reduce its ratio. However, the frequency of item A can hardly be reduced since it is equal to 1, so instead we increase the frequency of items B and C to 2. This new solution corresponding to a new set of frequencies (1, 2, 2) and a new calculated cycle time T^* that minimises the total cost, it is presented in Table 4 and illustrated in Fig. 2.

With different frequencies for different items the total cost function (Eq. (9)) becomes

$$C(\mathbf{f}, T) = \sum_i \left(\frac{f_i A_i}{T} + h_i \frac{d_i T}{2 f_i} (1 - o_i d_i) \right). \tag{14}$$

Notice that $q_i = d_i T / f_i$, and that the first part in Eq. (14) is the total set-up cost and the second part is the total inventory holding cost. We assume that the inventories are replenished immediately after they become empty. With different frequencies for different items the cycle time, T, must be large enough to cover the set-up and operation times for all items of the demand during the cycle

$$\sum_i (f_i s_i + o_i d_i T) \le T. \tag{15}$$

Therefore, the shortest possible time where the expected demand rates can be satisfied is

$$T_{\min} = \frac{\sum_i f_i s_i}{1 - \sum_i o_i d_i} = 1.35 \tag{16}$$

and consequently the time corresponding to the lowest cost, with given frequencies, is

$$T^* = \max\left(\frac{2\sum_i f_i A_i}{\sum_i h_i d_i (1 - o_i d_i)/f_i}, T_{\min} \right) = 22.48. \tag{17}$$

The total cost is 311.37 money units per day, which is 9.27 money units less than the total cost of the common cycle solution. According to Bomberger's restriction, the basic period must be

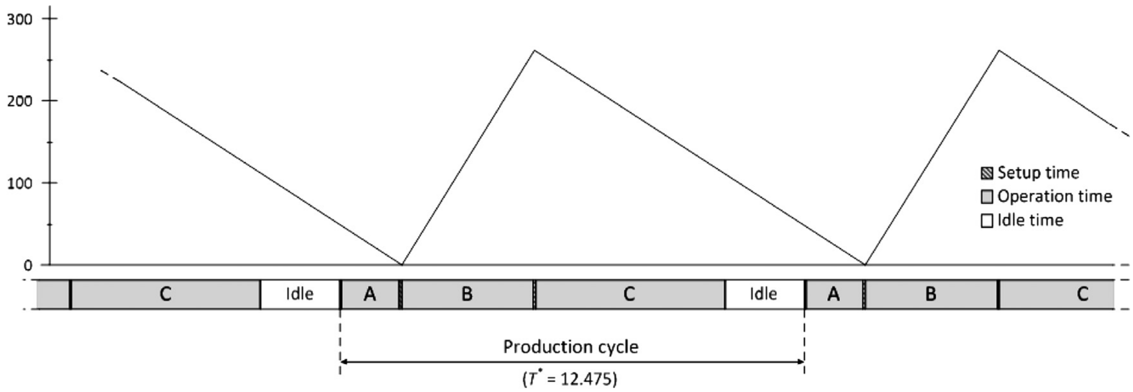

Fig. 1. The common cycle solution; time schedule and the inventory of item *B*.

Table 4
The basic period solution to the numerical example.

		A	*B*	*C*	Σ
(1)	Frequency: f_i	1	2	2	
(2)	Set-up time: $f_i\,s_i$	0.036	0.143	0.071	0.250
(3)	Operation time: $o_i d_i T^*$	2.810	6.423	9.100	18.333
(4)	(2)+(3)	2.846	6.566	9.171	18.583
(5)	Set-up cost/day	22.24	44.48	88.96	155.69
(6)	Inv. holding cost/day	22.45	83.78	49.45	155.69
(7)	(5)+(6)				311.37
(8)	$q_i^* = d_i T^*/f_i$	236.1	337.2	191.1	
(9)	Ratio: (5)/(6)	0.99	0.53	1.80	
(10)	1/Ratio: 1/(9)	1.01	1.88	0.56	

large enough to accommodate production of all items once. Therefore, the basic period, $T^*/\hat{f}$, must satisfy

$$\frac{T^*}{\hat{f}} \geq \sum_i \left(s_i + \frac{o_i d_i T^*}{f_i} \right) \tag{18}$$

where $\hat{f} = \max_i(f_i)$. In this case the basic period is $22.48/2 = 11.24$, and the time for producing all items once is $2.846 + 6.566/2 + 9.171/2 \approx 10.71$. Thus, all items can be produced during the basic period and the restriction is fulfilled. Bomberger solved the problem with dynamic programming; first finding a basic period and then deciding the frequency of each item. Unlike Bomberger's method, our method here first decides the frequency of each item and then calculates the optimal cycle time. Hence, the basic period is a result of the chosen frequencies and the calculated optimal cycle time.

6. The extended basic period approach

According to Table 4 and to the largest ratio or inverted ratio, it might be preferable to increase the frequency of item *B*. However, if we increase the frequency of item *B* one step (i.e. $f_B = 4$, using a power-of-two policy justified in forthcoming Section 7) $T_{\min}$ becomes 2.13 (Eq. (16)) and T^* becomes 29.816 (Eq. (17)), and the basic period, $T^*/4 \approx 7.46$, will not fulfil Eq. (18). The solution in Table 4 is the best basic period solution our method can find.

To find a better solution we need to consider another approach. In the *extended basic period approach* Bomberger's restriction on the basic period is relaxed so that the "period" only must cover the *average* set-up times and operation times of all items (Stankard and Gupta, 1969; Haessler and Hogue, 1976; Doll and Whybark, 1973; Elmaghraby, 1978). Thus

$$\frac{T^*}{\hat{f}} \geq \frac{\sum_i (f_i s_i + o_i d_i T^*)}{\hat{f}}. \tag{19}$$

By reworking Eq. (19) we end up with the same restriction on the cycle time as we presented in Eq. (15). But Eq. (19) is, in contrast to Eq. (18), satisfactory even if we increase the frequency of item *B* to "4" as we discussed earlier. Hence, the solution that was infeasible with the basic period approach is now feasible. The extended basic period solution is presented in Table 5.

However, relaxing the period length restriction gives us a new problem. The production of item *A* requires 3.763 days, item *B* requires $8.805/4 = 2.201$ days and item *C* requires $12.140/2 = 6.070$ days. In two of the four periods both item *B* and item *C* needs to be scheduled and therefore those periods must be at least $2.201 + 6.070 = 8.271$ days, which is longer than $T^*/4 \approx 7.45$. Thus, all four periods cannot be equally long. When we cannot schedule the production in equally long periods the traditional inventory holding cost approximation and the cost calculations in Table 5 does not hold (cf. Nilsson and Segerstedt (2008)). The inequalities force the production of item *B* and item *C* to start before the inventories reach zero to avoid shortages. Therefore an extra inventory holding cost is created, and hence the real total cost is more than the 301.85 money units shown in Table 5.

How to calculate the extra cost is described in Holmbom et al. (2013). First, a detailed production schedule is created. The scheduling procedure aims to make the four periods as equal as possible. If all periods have the same length there is no extra inventory and the preliminary cost estimation in Table 5 is correct. Second, the potential "early starts" are calculated. The early starts depend on the difference between the actual period length and the theoretical period length if all periods would have been equal. The schedule of the solution in Table 5 is presented in Table 6 and illustrated in Fig. 3. Early starts are necessary for item *B* in periods 1 and 3.

With the extra inventory the total cost becomes (cf. Holmbom et al. (2013))

$$C(\mathbf{f}, T^*) = \sum_i \left(\frac{f_i A_i}{T^*} + h_i \frac{d_i T^*}{2 f_i} (1 - o_i d_i) + h_i \sum_{j=1}^{\hat{f}} \frac{d_i t_{ij}}{f_i} \right) = 310.37. \tag{20}$$

The extra inventory cost is $\sum_i h_i \sum_j (d_i t_{ij}/f_i) = 8.53$ money units per day. Observe that the machine is idle during 5.109 days of the total 29.816 days to keep the inventory levels stationary over time. Hence, the machine is idle during 17% of the total time; the same as for the common cycle solution. The total cost is 1.00 money units less than the cost of the basic period solution, and 10.27 money units less than the cost of the common cycle solution. This is the best extended basic period solution that can be found with the current method.

The set-up times in the numerical example are relatively short. If they are multiplied with a factor 4 to 2 h for item *A* and *C* and 4 h for item *B*, the schedule would be like Table 7.

The "optimal" cycle time and the economical frequencies are still the same, since the costs are unchanged. The idle time and early

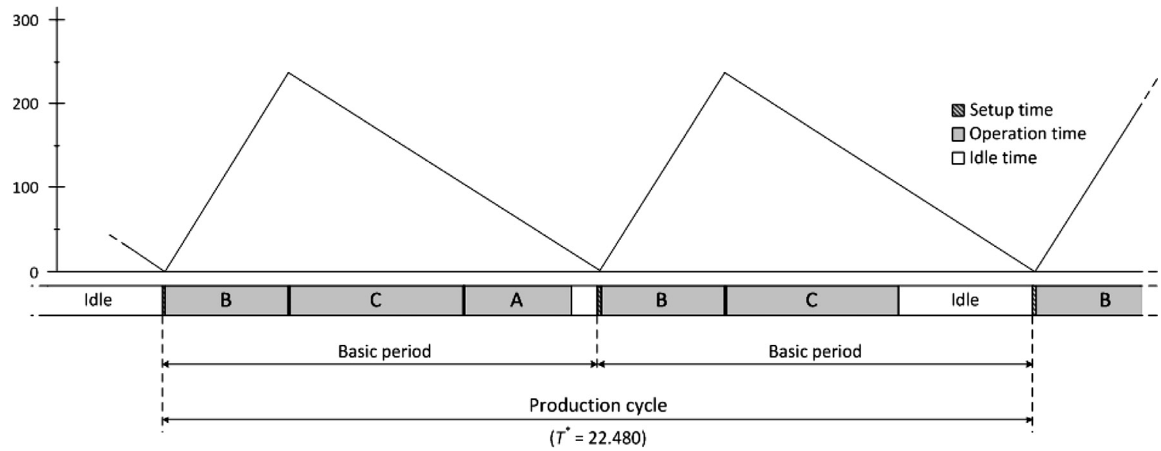

Fig. 2. The basic period solution; time schedule and the inventory of item *B*. The items are scheduled in descending order according to (1) frequency and (2) production time; beginning with the item that has highest frequency and longest production time.

Table 5
The extended basic period solution to the numerical example.

		A	*B*	*C*	Σ
(1)	Frequency: f_i	1	4	2	
(2)	Set-up time: $f_i\, s_i$	0.036	0.286	0.071	0.036
(3)	Operation time: $o_i d_i T^*$	3.727	8.519	12.069	24.315
(4)	(2)+(3)	3.763	8.805	12.140	24.708
(5)	Set-up cost/day	29.78	55.56	65.59	150.92
(6)	Inv. holding cost/day	16.77	67.08	67.08	150.92
(7)	(5)+(6)				301.85
(8)	$q_i^* = d_i T^*/f_i$	313.1	223.6	253.4	
(9)	Ratio: (5)/(6)	1.78	0.83	0.98	
(10)	1/Ratio: 1/(9)	0.56	1.21	1.02	

Table 6
The schedule of the extended basic period solution.

	Period 1	Early start	Period 2	Early start	Period 3	Early start	Period 4	Early start
Production *B*	2.201	0.817	2.201	–	2.201	0.817	2.201	–
Production *C*	6.070	–			6.070	–		
Production *A*			3.763	–				
Idle:	0.000 8.271		0.673 6.637		0.000 8.271		4.436 6.637	
Cumulated:								
Actual length	8.271		14.908		23.179		29.816	
Equal length	7.454		14.908		22.362		29.816	

starts have changed and the total cost is now $301.85+2\times0.696$ $(30\times1.115/4)=313.49$. The machine is idle during 13% of the time, and in the common cycle solution of this modified example the idle time is 14%.

If we assume a complementary facility cost of 300 money units per hour (5 MU/min, 4200 MU/day) independent of the machine is idle or not, the overall total cost of the common cycle solution increases to $(12.475\times4200+12.475\times4200+12.475\times320.64)/12.475=4200+320.64=4520.64$ MU/day. For the extended basic period solution the increase is $4200+310.37=4510.37$ MU/day; still a better solution than the common cycle solution. The shift factor, *K*, is the variable that creates costs. Hence, a significant cost reduction can be achieved if it is possible to produce the demand in fewer hours and thus in fewer shifts per day. In cases where the idle time is large, it is financially beneficial to produce the demand with a minor *K* given that T_{inf} is larger than zero but not too large (approximately <15). The shift factor will not influence the frequencies; the economical frequencies only depend on the set-up costs and inventory holding costs.

7. The Bomberger problem

Due to the importance of the Bomberger problem instance in this research field, we will here present a scheduled solution to it. The Bomberger problem is presented in Table 8.

We have inverted the production rates in the Bomberger problem to "capacity" ($o_i = 1/\kappa_i$). The holding cost per unit and day, h_i, is calculated from an original labour and material cost per unit, c_i, multiplied with 0.1 (inventory interest rate) and divided by 240 (days per year) (cf. Bomberger (1966)). Unfortunately, the original article contains a printing mistake where c_2 is presented as 0.1175 instead of 0.1775. The common cycle solution that Bomberger presents in the same article, $T=41.17$, can only be replicated if $c_2=0.1775$, so it is surely a printing mistake. In the early 1970s the printing mistake must have been known, but it seems like no one commented upon it. The first comment may be in Cooke et al. (2004) but unfortunately some earlier published articles are based on the wrong numbers. (Segerstedt (1999) contains another printing mistake concerning h_i.)

Table 9 shows the best known solution to the Bomberger problem found by Doll and Whybark (1973), Goyal (1975) and Segerstedt (1999).

The Bomberger problem is possible to schedule in 8 equal periods, see Table 10, and thus we do not need to consider any early starts or extra inventory.

In the Bomberger problem the utilisation is high and the machine is idle only during 4.6% of the time. In that respect it is quite surprising that the production of all items fit into 8 equal periods. It can perhaps partly be explained by the fact that there are many different items to schedule, and hence many small pieces of the puzzle to put together. High utilisation is preferred from a total cost perspective, but too high utilisation will transform the process to a bottleneck (cf. Hopp (2011)) and create costs such as queuing, delays and more work in process (WIP). Thus, too high utilisation must be avoided to facilitate short delivery times, high delivery precision and short throughput times.

The schedule in Table 10 cannot be found if we apply the scheduling rules of the Holmbom et al. (2013) method strictly. If we follow that allocation method strictly, we end up with a

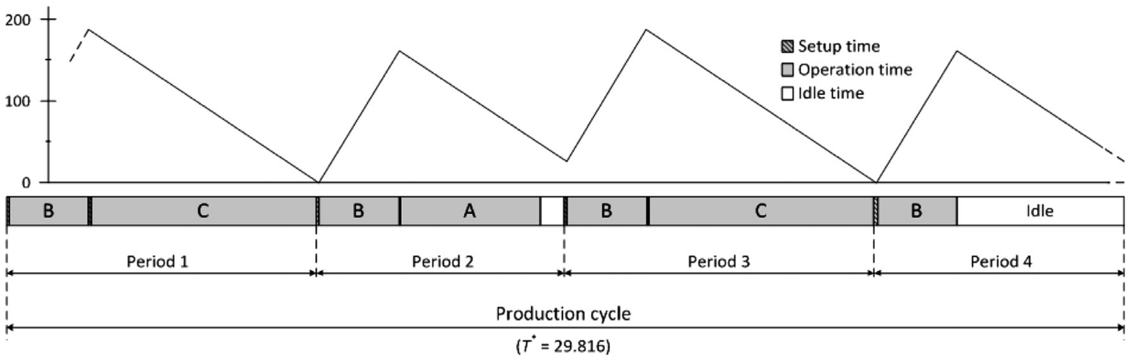

Fig. 3. The extended basic period solution; time schedule and the inventory of *B*.

Table 7
The extended basic period solution; with longer set-up times.

	Period 1	Early start	Period 2	Early start	Period 3	Early start	Period 4	Early start
Production *B*	2.392	1.115	2.392	–	2.392	1.115	2.392	–
Production *C*	6.177	–			6.177	–		
Production *A*			3.870	–				
Idle:	0.000		0.077		0.000		3.947	
	8.569		6.637		8.569		6.637	
Cumulated:								
Actual length	8.569		14.908		23.477		29.816	
Equal length	7.454		14.908		22.362		29.816	

Table 8
The Bomberger problem (time unit: days).

Item i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
d_i	400	400	800	1600	80	80	24	340	340	400
κ_i	30,000	8000	9500	7500	2000	6000	2400	1300	2000	15,000
o_i	3.333×10^{-5}	1.250×10^{-4}	1.053×10^{-4}	1.333×10^{-4}	5.000×10^{-4}	1.667×10^{-4}	4.167×10^{-4}	7.692×10^{-4}	5.000×10^{-4}	6.667×10^{-5}
S_i	0.125	0.125	0.25	0.125	0.50	0.25	1	0.5	0.75	0.125
A_i	15	20	30	10	110	50	310	130	200	5
c_i	0.0065	0.1775	0.1275	0.1000	2.7850	0.2675	1.5000	5.9000	0.9000	0.0400
h_i	2.708×10^{-6}	7.396×10^{-5}	5.313×10^{-5}	4.167×10^{-5}	1.160×10^{-3}	1.115×10^{-4}	6.250×10^{-4}	2.458×10^{-3}	3.750×10^{-4}	1.667×10^{-5}

slightly different schedule where all periods are not equal. This is because the method suggests that *item 10* should be scheduled in periods 1, 3, 5 and 7 instead of periods 2, 4, 6 and 8, which is required to achieve period equality. The allocation method can be modified with complementary scheduling rules to find the exact schedule shown in Table 10. (From a practical point of view the cost difference is negligible. If we put *item 10* in periods 1, 3, 5 and 7, then period 4 contains 23.66 days of production, which in turn force *items 8* and *4* to start 0.24 days early. The total cost per day increases from 32.07 to 32.10; +0.09%.)

After Bomberger's publication many researchers and authors have made different contributions to the ELSP area. Some well-known references, some already mentioned, are Doll and Whybark (1973), Goyal (1973, 1975), Haessler and Hogue (1976), Elmaghraby (1978), Hsu (1983), Axsäter (1987), Zipkin (1991), Gallego and Roundy (1992) and Bourland and Yano (1997). Among the more recent publications are Khoury et al. (2001), Soman et al. (2004), Cooke et al. (2004) and Yao (2005).

Elmaghraby (1978) presents an overview of earlier research. Lopez and Kingsman (1991) make a review and compare different solution methods. They argue that the "power-of-two policy", of the basic period, is a requirement for achieving schedule feasibility in practice. Yao and Elmaghraby (2001) also show that the power-of-two policy simplifies the construction of feasible cyclic schedules. A recent review is made by Chan et al. (2013) who summarise the ELSP research during the last 15 years (1997–2012).

Remanufacturing is a production and problem area where ELSP is also applied; e.g. Tang and Teunter (2006), Teunter et al. (2009) and Zanoni et al. (2012). Zanoni et al. (2012) use the same principle as shown in this article; the ratio between the set-up cost and the inventory holding cost of each item should be as close as possible to "1". Segerstedt (2004) shows that the same principle can be extended to several machines and multi-level production. The same principle has also been applied to other problems than ELSP with good results, e.g. the Joint Replenishment Problem (Nilsson et al., 2007) and the One Warehouse N-retailer Problem (Abdul-Jalbar et al., 2010).

8. Summary and practical implications

The aim of this article was to describe the development of the ELSP field from the EOQ formula to the solution methods of today. We have discussed the early history of EOQ and EPL and showed the related formulas. We have also discussed the limitations of these formulas when many different items are produced in the same machine. In such situations the EOQ and EPL formulas do not guarantee that the order quantities will cover demand, or the

Table 9
The best known solution to the Bomberger problem.

Item i	1	2	3	4	5	6	7	8	9	10	
f_i	1	4	4	8	4	2	1	8	4	4	
T^*											187.40
$s_i+o_id_iT^*/f_i$	2.623	2.468	4.197	5.121	2.374	1.500	2.874	6.626	8.715	1.374	
f_iA_i/T^*	0.08	0.43	0.64	0.43	2.35	0.53	1.65	5.55	4.27	0.11	16.04
$h_i(1-o_id_i)d_iT^*/(2f_i)$	0.10	0.66	0.91	0.61	2.09	0.41	1.39	7.23	2.48	0.15	16.03
											32.07

Table 10
The extended basic period solution to the Bomberger problem.

Item	Period								
	1	2	3	4	5	6	7	8	
8	6.626	6.626	6.626	6.626	6.626	6.626	6.626	6.626	
4	5.121	5.121	5.121	5.121	5.121	5.121	5.121	5.121	
9	8.715		8.715		8.715		8.715		
3		4.197		4.197		4.197		4.197	
2		2.468		2.468		2.468		2.468	
5		2.374		2.374		2.374		2.374	
10		1.374		1.374		1.374		1.374	
6	1.500				1.500				
7			2.874						
1							2.498		
Σ	21.962	22.160	23.336	22.160	21.962	22.160	22.960	22.160	178.86
Idle	1.463	1.265	0.089	1.265	1.463	1.265	0.465	1.265	8.54
ΣΣ	23.425	23.425	23.425	23.425	23.425	23.425	23.425	23.425	187.40

capacity will be enough, during the total time of both production and set-up. Therefore, we had to consider the items jointly and we introduced the common cycle solution. With the common cycle solution we computed a feasible production schedule for a small numerical example with three items. This production schedule was further improved when we used the basic period approach to solve the example. The basic period approach allows the items to be produced with different frequencies, which often can be cost-effective. In our example the basic period approach reduced the total cost per day from 320.64 (the common cycle solution) to 311.37 money units/day. This cost was even further reduced when we applied the extended basic period approach. In the extended basic period approach the restriction on the "period" is relaxed, which can enable better utilisation of the capacity and less idle time. However, with this relaxed restriction there is no guarantee that the periods will be equally long and therefore the production of some items must start before the inventory is empty, which will lead to a higher average inventory. Hence, the cost calculation depends on the scheduling and requires a solution method that creates a detailed production schedule. The cost of the extended basic period approach solution was shown to be 310.37 money units, and not 301.85 as the traditional inventory holding cost approximation suggested; a significant difference that previous ELSP literature has paid little attention to. But despite that extra inventory holding cost, the extended basic period solution came closer to the lower bound solution on 299.40 (derived from the EPL formula) then the common cycle solution and the basic period solution.

In a practical situation it is sensible to have a production schedule that is repeated at regular time intervals to create a production pattern. In the solution of the Bomberger problem the schedule would be repeated every 187.40 day, which is a bit awkward and cumbersome time frame. The production of *items 7* and *3* would almost cover the demand of a whole year, and if the demand changes over time there is a risk for obsolete overstock. Therefore a too long cycle time should be avoided. The method of Holmbom et al. (2013) is easy to adapt to different time intervals. In practical situations the scheduling may be done explicitly for the items with the highest recommended frequencies, while items with low demand and frequency (produced once per sixth month or less) can be scheduled in a buffer every period.

The set-up time is sometimes dependent of the sequence of the production; e.g. die-casting of plastic products (the set-up time is shorter if the item has the same colour as the previous produced item) or manufacturing of pinions and gears (the set-up time is shorter if the item has the same module or the same size of the gear tooth as the previous item). The model presented here does not explicitly consider sequence dependent set-up times, but it can be handled by creating a schedule based on the average set-up time.

In a practical situation everything is not deterministic; set-up times and operation times are often stable but expected demand rates can change. Therefore the ELSP analysis and calculation must be done regularly. Brander et al. (2005) show that deterministic models can be used even if the demand is stochastic. They emphasise the importance of rules to decide whether the production should start or not. Levén and Segerstedt (2007), with inspiration from Leachman and Gascon (1988), further develop such decision rules. For example, a decision rule can be based on the current inventory; if the inventory of an item does not cover expected demand until the next possibility to produce the item, it should be produced in the current period. The cover time, i_i/d_i, where i_i is the current inventory of item i, can be used to prioritise the items to avoid shortages.

Time-varying lot sizes have been studied and suggested by e.g. Dobson (1987) and Moon et al. (2002). However, even though time-varying lot sizes work satisfactory in theory there may be considerable practical disadvantages such as variations in the material supply and inventory floor space. Time-varying lot sizes are not compatible with philosophies such as *lean production* that strives to eliminate all kinds of variation and promotes standardised working routines. This may be an explanation to the larger interest for fixed order quantities. In the literature review by Chan et al. (2013) on recent research trends of ELSP only 11% of the articles dealt with time-varying lot sizes.

Reports of ELSP implementations in practice are rare, but van den Broecke et al. (2005, 2008) report about successful applications of a variant of the method presented by Doll and Whybark (1973), and Taj et al. (2012) report about successful applications of a model similar to Segerstedt (1999).

Hopefully this article can be used to quickly introduce master and graduate students to the practical and theoretical problem ELSP found in countless industrial processes.

Acknowledgements

The authors are very thankful for valuable comments from anonymous reviewers. This work has been supported by Kolarctic Project, Barents Logistics 2; with complement financing from County Administrative Board of Norrbotten, Region Västerbotten and Luleå City.

References

Abdul-Jalbar Betancor, B., Segerstedt, A., Sicilia, J., Nilsson, A., 2010. A new heuristic to solve the one-warehouse N-retailer problem. Comput. Oper. Res. 37 (2), 265–272.
Alford, L.P. (Ed.), 1934. Cost and Production Handbook. Ronald Press, New York.
Axsäter, S., 1987. An extension of the extended basic period approach for economic lot scheduling problems. J. Optim. Theory Appl. 52 (2), 179–189.
Bomberger, E., 1966. A dynamic programming approach to a lot size scheduling problem. Manag. Sci. 12 (11), 778–784.
Bourland, K.E., Yano, C.A., 1997. A comparison of solution approaches for the fixed sequence economic lot scheduling problem. IIE Trans. 29 (2), 103–108.
Brander, P., 2006. Inventory Control and Scheduling Problems in a Single-Machine Multi-Item System (Doctoral thesis). Luleå University of Technology.
Brander, P., Levén, E., Segerstedt, A., 2005. Lot sizes in a capacity constrained facility – a simulation study of stationary stochastic demand. Int. J. Prod. Econ. 93–94, 375–386.
Brander, P., Segerstedt, A., 2009. Economic lot scheduling problems incorporating a cost of using the production facility. Int. J. Prod. Res. 47 (13), 3611–3624.
Buffa, E.S., 1969. Modern Production Management, 3rd ed. John Wiley & Sons, New York.
van den Broecke, F., van Landeghem, H., Aghezzaf, E.-H., 2005. An application of cyclical master production scheduling in a multi-stage, multi-product environment. Prod. Plan. Control 16 (8), 796–809.
van den Broecke, F., van Landeghem, H., Aghezzaf, E.-H., 2008. Implementing a near-optimal solution for the multi-stage, multi-product capacitated lot-sizing problem by rolling out a cyclical production plan. Int. J. Prod. Econ. 112 (1), 121–137.
Camp, W.E., 1922. Determining the production order quantity. Manag. Eng. 2, 17–18.
Chan, H.K., Chung, S.H., Lim, M.K., 2013. Recent research trend of economic-lot scheduling problems. J. Manuf. Technol. Manag. 24 (3), 465–482.
Cooke, D.L., Rohleder, T.R., Silver, E.A., 2004. Finding effective schedules for the economic lot scheduling problem: a simple mixed integer programming approach. Int. J. Prod. Res. 42 (1), 21–36.
Dobson, G., 1987. The economic lot-scheduling problem: achieving feasibility using time-varying lot sizes. Oper. Res. 35 (5), 764–771.
Doll, C.L., Whybark, D.C., 1973. An iterative procedure for the single-machine multi-product lot scheduling problem. Manag. Sci. 20 (1), 50–55.
Elmaghraby, S.E., 1978. The Economic Lot Scheduling Problem (ELSP): review and extensions. Manag. Sci. 24 (6), 587–598.
Erlenkotter, D., 1990. Ford Whitman Harris and the Economic Order Quantity Model. Oper. Res. 38 (6), 937–946.
Gallego, G., Roundy, R., 1992. The economic lot scheduling problem with finite backorder costs. Nav. Res. Logist. 39, 729–739.
Gallego, G., Shaw, D.X., 1997. Complexity of the ELSP with general cyclic schedules. IIE Trans. 29 (2), 109–113.
Goyal, S.K., 1973. Scheduling a machine multi-product system. Oper. Res. Q. 24 (2), 261–269.
Goyal, S.K., 1975. Scheduling a single machine multi-product system: a new approach. Int. J. Prod. Res. 13 (5), 487–493.
Grubbstrom, R.W., 1995. Modelling production opportunities – an historical overview. Int. J. Prod. Econ. 41 (1–3), 1–14.
Hadley, G., Whitin, T.M., 1963. Analysis of Inventory Systems. Prentice Hall, Englewood Cliffs, N. J.
Hanssmann, F., 1962. Operations Research in Production and Inventory Control. John Wiley and Sons, New York.
Haessler, R.W., Hogue, S.L., 1976. A note on the single-machine multi-product lot scheduling problem. Manag. Sci. 22 (8), 909–912.
Harris, F.W., 1913. How many parts to make at once. Fact. Mag. Manag. 10 (2), 135–136 (152).
Hopp, W.J., 2011. Supply Chain Science. Waveland Press, Long Grove, IL.
Holmbom, M., Segerstedt, A., Sluis, E., 2013. A solution procedure for Economic Lot Scheduling Problems even in high utilisation facilities. Int. J. Prod. Res. 51 (12), 3765–3777.
Hsu, W.-L., 1983. On the General Feasibility Test of Scheduling Lot Sizes for Several Products on One Machine. Management Science 29 (1), 93–105.
Khoury, B.N., Abboud, N.E., Tannous, M.M., 2001. The common cycle approach to the ELSP problem with insufficient capacity. Int. J. Prod. Econ. 73 (2), 189–199.
Leachman, R.C., Gascon, A., 1988. A heuristic scheduling policy for multi-item, single-machine production systems with timevarying, stochastic demands. Manag. Sci. 34 (3), 377–390.
Levén, E., Segerstedt, A., 2007. A scheduling policy for adjusting economic lot quantities to a feasible solution. Eur. J. Oper. Res. 179 (2), 414–423.
Lopez, M.A. N., Kingsman, B.G., 1991. The Economic Lot Scheduling Problem: theory and practice. Int. J. Prod. Econ. 23 (1–3), 147–164.
Maynard, H.B. (Ed.), 1956. Industrial Engineering Handbook, 1st ed. McGraw-Hill, New York.
Moon, I., Silver, E.A., Choi, S., 2002. Hybrid genetic algorithm for the economic lot-scheduling problem. Int. J. Prod. Res. 40, 809–824.
Nahmias, S., 2009. Production and Operations Analysis, 6th ed., McGraw Hill, New York.
Nilsson, A., Segerstedt, A., van der Sluis, E., 2007. A new iterative heuristic to solve the Joint Replenishment Problem using a spread-sheet technique. Int. J. Prod. Econ. 108 (1–2), 399–405.
Nilsson, K., Segerstedt, A., 2008. Corrections of costs to feasible solutions of Economic Lot Scheduling Problems. Comput. Ind. Eng. 54 (1), 155–168.
Rogers, J., 1958. A computational approach to the economic lot scheduling problem. Manag. Sci. 4 (3), 264–291.
Silver, E.A., Pyke, D.F., Peterson, R., 1998. Inventory Management and Production Planning and Scheduling. John Wiley, New York.
Soman, C.A., van Donk, D.P., Gaalman, G.J.C., 2004. A basic period approach to the economic lot scheduling problem with shelf life considerations. Int. J. Prod. Res. 42 (8), 1677–1689.
Segerstedt, A., 1999. Lot sizes in a capacity constrained facility with available initial inventories. Int. J. Prod. Econ. 59 (1–3), 469–475.
Segerstedt, A., 2004. Frequency approach for treating capacity-constrained multi-level production. Int. J. Prod. Res. 42 (16), 3119–3137.
Segerstedt, A., 2009. Logistik med fokus på material- och produktionsstyrning (Logistics with Focus on Material- and Production Control). Liber, Malmo (in Swedish).
Stankard Jr., M.F., Gupta, S.K., 1969. A note on Bomberger's approach to lot size scheduling: heuristic proposed. Manag. Sci. 15 (7), 449–452.
Taft, E.W., 1918. The most economical production lot. Iron Age 101, 1410–1412.
Taj, S., Nedeltcheva, G.N., Pfeil, G., Roumaya, M., 2012. A spread-sheet model for efficient production and scheduling of a manufacturing line/cell. Int. J. Prod. Res. 50 (4), 1141–1154.
Tang, O., Teunter, R., 2006. Economic lot scheduling problem with returns. Prod. Oper. Manag. 15 (4), 488–497.
Teunter, R., Tang, O., Kaparis, K., 2009. Heuristics for the economic lot scheduling problem with returns. Int. J. Prod. Econ. 118 (1), 323–330.
Yao, M.-J., 2005. The economic lot scheduling problem without capacity constraints. Ann. Oper. Res. 133 (1–4), 193–205.
Yao, M.-J., Elmaghraby, S.E., 2001. The economic lot scheduling problem under power-of-two policy. Comput. Math. Appl. 41 (10–11), 1379–1393.
Wilson, R.H., 1934. A scientific routine for stock control. Harv. Bus. Rev. 13, 116–128.
Zanoni, S., Segerstedt, A., Tang, O., Mazzoldi, L., 2012. Multi-product economic lot scheduling problem with manufacturing and remanufacturing using a basic period policy. Comput. Ind. Eng. 62 (4), 1025–1033.
Zipkin, P., 1991. Computing optimal lot sizes in the economic lot scheduling problem. Oper. Res. 39 (1), 56–63.